

Local supersymmetric extensions of the Poincaré and AdS invariant gravity

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ABSTRACT: In all the odd dimensions which allow Majorana spinors, we consider a gravitational Lagrangian possessing local Poincaré invariance and given by the dimensional continuation of the Euler density in one dimension less. We show that the local supersymmetric extension of this Lagrangian requires the algebra to be the maximal extension of the $\mathcal{N} = 1$ super-Poincaré algebra. By maximal, we mean that in the right hand side of the anticommutator of the Majorana super charge appear all the possible “central charges”. The resulting action defines a Chern-Simons gauge theory for the maximal extension of the super-Poincaré algebra. In these dimensions, we address the same problem for the AdS invariant gravity and we derive its supersymmetric extension for the minimal super-AdS algebra. The connection between both models is realized at the algebraic level through an expansion of their corresponding Lie super algebras. Within a procedure consistent with the expansion of the algebras, the local supersymmetric extension of the Poincaré invariant gravity Lagrangian is derived from the super AdS one.

KEYWORDS: Gauge Symmetry, Chern-Simons Theories, Supersymmetric gauge theory.

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1. Introduction

Two of the main fundamental assumptions in general relativity are the requirements of general covariance and the fact that the field equations for the metric are of second order. In view of this, in three and four dimensions it is natural to describe the spacetime geometry by the Einstein-Hilbert action (with or without a cosmological constant) while for dimensions greater than four, a more general theory can be used. This fact has been first noticed by Lanczos in five dimensions [1] and later generalized by Lovelock for any dimensions d [2]. The resulting theory is described by the Lovelock Lagrangian, which is a d -form constructed with the vielbein e^a , the spin connection ω^{ab} , and their exterior derivatives without using the Hodge dual. The corresponding action contains the same degrees of freedom than those in the Einstein-Hilbert action [3] and is the most general low-energy effective theory of gravity derived from string theory [4]. The Lagrangian is a polynomial of degree $[d/2]$ in the curvature two-form, $R^{ab} = d\omega^{ab} + \omega_c^a \wedge \omega^{cb}$,

$$\mathcal{L}^{(d)} = \sum_{p=0}^{[d/2]} \alpha_p \epsilon_{a_1 \dots a_d} R^{a_1 a_2} \dots R^{a_{2p-1} a_{2p}} e^{a_{2p+1}} \dots e^{a_d}, \quad (1.1)$$

where the α_p are arbitrary dimensionful coupling constants and where wedge products between forms are understood. The first two terms in (1.1) are nothing but the cosmological constant and the Einstein-Hilbert pieces. The action is invariant under the local Lorentz rotations by construction. Interestingly enough, in odd dimensions $d = 2n + 1$, there is a particular choice of the coefficients α_p that allows to extend the local Lorentz symmetry into a local (anti) de Sitter or Poincaré symmetry. The latter choice is simply achieved by choosing $\alpha_p = \delta_p^n$. The resulting gravitational Lagrangian \mathcal{L}_P corresponds to the dimensional continuation of the Euler density and is given by

$$\mathcal{L}_P = \epsilon_{a_1 \dots a_{2n+1}} R^{a_1 a_2} \dots R^{a_{2n-1} a_{2n}} e^{a_{2n+1}}. \quad (1.2)$$

Note that \mathcal{L}_P reduces to the Einstein-Hilbert action only in three dimensions. The invariance of the gravitational Lagrangian \mathcal{L}_P under the local Poincaré translations whose action on the dynamical fields is given by

$$\delta e^a = D\lambda^a := d\lambda^a + \omega_b^a \lambda^b, \quad \delta \omega^{ab} = 0, \quad (1.3)$$

is a direct consequence of the Bianchi identity. In what follows, we refer to the gravitational Lagrangian (1.2) as the Poincaré invariant gravity Lagrangian.

Let us stress an important feature that will be our guiding principle in the construction of the supersymmetric extension of the Lagrangian \mathcal{L}_P . The gravitational Lagrangian (1.2) belongs to the class of Chern-Simons gauge theories with Yang-Mills gauge symmetries. Indeed, the local Poincaré translations (1.3) correspond to a gauge transformation

$$\delta_\lambda A = d\lambda + [A, \lambda], \quad (1.4)$$

with parameter $\lambda = \lambda^a P_a$. Note that we have parameterized the components of the gauge field A in the adjoint representation of the Poincaré group with generators P_a and J_{ab} as

$$A = \frac{1}{2} \omega^{ab} J_{ab} + e^a P_a. \quad (1.5)$$

In addition, the Lagrangian \mathcal{L}_P is a Chern-Simons form for the Poincaré connection (1.5) since its exterior derivative satisfies

$$d\mathcal{L}_P = \langle F^{n+1} \rangle. \quad (1.6)$$

Here F is the field strength associated to the Poincaré connection (1.5) and the symbol $\langle \dots \rangle$ refers to a symmetric invariant $(n+1)$ -tensor on the Poincaré group whose only nonvanishing component is given by

$$\langle J_{a_1 a_2}, \dots, J_{a_{2n-1} a_{2n}}, P_{a_{2n+1}} \rangle = \frac{2^n}{n+1} \epsilon_{a_1 \dots a_{2n+1}}. \quad (1.7)$$

In three dimensions the Lagrangian (1.2) corresponds to the Einstein-Hilbert Lagrangian and the questions relative to its supersymmetric extension have already been answered in the past. In this case, the supersymmetry is easily achieved by introducing only a spinor field and the resulting action can be viewed as a gauge theory for the super-Poincaré algebra [5]. Later on, the local supersymmetric extension of the gravitational Lagrangian (1.2) has been constructed in arbitrary odd dimensions [6]. The resulting supersymmetric theories possess a rich geometrical structure encoded by a fibre bundle structure and by the fact that the supersymmetry closes off-shell without requiring the introduction of auxiliary fields. In a generic odd dimension, the supersymmetric extension of the Poincaré invariant gravity Lagrangian defines a Chern-Simons gauge theory with gauge group identified with the super five brane algebra [6]. This algebra is a nontrivial extension of the Poincaré algebra which is spanned by the Poincaré generators together with a 5-form “central charges” $Z_{a_1 \dots a_5}$ and complex fermionic generators Q^α and \bar{Q}_α whose anticommutator reads

$$\{Q^\alpha, \bar{Q}_\beta\} = -i(\Gamma^a)^\alpha_\beta P_a - \frac{i}{5!} (\Gamma^{a_1 \dots a_5})^\alpha_\beta Z_{a_1 \dots a_5}. \quad (1.8)$$

Note that the bosonic generator $Z_{a_1 \dots a_5}$ required by supersymmetry is truly a central charge only in five dimensions while in odd dimensions $d > 5$, it does not commute with the Lorentz rotation generator because of its Lorentz indices. For some reviews on Chern-Simons supergravity, see e.g. [7].

In view of the work of ref. [6], the supersymmetric extension of the Poincaré invariant gravity (1.2) seems to confer a particular status on the super five brane algebra (1.8). However, in nine dimensions it has been shown that the super-Poincaré algebra with a $U(1)$ central extension can accommodate such construction [8] while in eleven dimensions a supersymmetric extension of the Poincaré invariant gravity (1.2) with Majorana spinors has been achieved for the M-algebra [9],

$$\{Q, Q\} = (C\Gamma^a)P_a + (C\Gamma^{ab})Z_{ab}^{(2)} + (C\Gamma^{a_1 \dots a_5})Z_{a_1 \dots a_5}^{(5)}. \quad (1.9)$$

Here $Z_{ab}^{(2)}$ and $Z_{a_1 \dots a_5}^{(5)}$ are “central charges” corresponding to the two types of extended objects that couple to the Abelian 3–form of standard eleven supergravity [10]. From an algebraic point of view, the M-algebra (1.9) corresponds to the maximal extension of the $\mathcal{N} = 1$ super-Poincaré algebra in the sense that in the right hand side of the anticommutator of the Majorana super charge appear all the possible “central charges” allowing by symmetry [11]. These two results suggest that the super five brane algebra does not fulfill all the possibilities. Another interesting observation concerns the dimensions 3 and 11 which precisely allow Majorana spinors. In fact, in three dimensions (resp. in eleven dimensions), the supersymmetric extension of the Poincaré invariant gravity Lagrangian can be constructed and the resulting action is a gauge theory for the maximal extension of the $\mathcal{N} = 1$ super-Poincaré algebra [5] (resp. in [9]). The purpose of this article is to extend these two results in all the odd dimensions allowing Majorana spinors $d = 3 \pmod{8}$. Indeed, we shall prove that the maximal extension of the $\mathcal{N} = 1$ super-Poincaré algebra is always compatible with the construction of the local supersymmetric extension of the Poincaré invariant gravity Lagrangian. In other words, this means that it is always possible to construct a local supersymmetric extension of the Lagrangian (1.2) such that the resulting action can be viewed as a gauge theory for the maximal extension of the $\mathcal{N} = 1$ super-Poincaré algebra. As shown below, the supersymmetric action can be written in a simple form as a trace and its invariance under supersymmetry is a direct consequence of a Fierz rearrangement.

In these particular dimensions, we also construct the local supersymmetric extensions of the AdS gravity for which the gauge group is identified with the minimal supersymmetric extension of the anti-de Sitter (AdS) group. By minimal we mean the smallest super algebra that contains the AdS generators.

It is well-known that the Poincaré algebra can be viewed as a Wigner-Inönü contraction of the (A)dS algebra. Since we are dealing with supersymmetric extensions of these algebras, it is natural to ask whether the maximal extension of the $\mathcal{N} = 1$ super-Poincaré algebra can be obtained from the smallest super algebra containing the AdS generators. In the current literature, there exist various ways to obtain a Lie algebra from a given one, as for example the Wigner-Inönü contraction. In general, these procedures are restrictive in the sense that the starting and resulting algebras have necessarily the same

dimension. In [12], de Azcárraga *et al.* have proposed a consistent way of generating a Lie algebra whose dimension is greater than the original one. This method originally considered by Hatsuda and Sakaguchi [13] in a less general context consists in expanding the Maurer-Cartan one-forms in powers of a real parameter in such way that the Maurer-Cartan equations are satisfied order by order, leading to a closed algebra at each order. Within this process called *expansion*, the M-algebra has been derived from the minimal extension of the eleven-dimensional AdS algebra $\mathfrak{osp}(32|1)$ that has 55 generators less [12]. In our case, we will show that the maximal extension of the $\mathcal{N} = 1$ super-Poincaré algebra can be obtained from the minimal extension of the AdS algebra through the expansion method. In addition, we also address the problem of the correspondence at the level of the supersymmetric actions that means finding a process compatible with the expansion of the Lie super algebras that permit to obtain the super Poincaré theory from the super AdS one. In this connection, Segal in [14], stated that if two physical theories are linked through a limiting process then there should also exist a corresponding limit between their underlying symmetry groups. In this optics and as a first step, we consider the standard Wigner-Inönü contraction which is the natural option to obtain an interesting theory at the vanishing cosmological constant limit. In this case, we show that although the Wigner-Inönü contraction gives rise to a consistent theory at the vanishing cosmological constant limit, the resulting action being decoupled from the vielbein and the gravitino is of little interest. In contrast, the next order in the contraction contains the gravitational Lagrangian (1.2) but does not define a supersymmetric Lagrangian. As we shall show below, the terms in the next order can be made supersymmetric by exploiting the possibility of adding to the spin connection a tensor under the Lorentz group. This splitting of the spin connection, which turns out to be equivalent to expanding the minimal extension of the AdS algebra, has important consequences. Indeed, within this process the gauge structure relative to the maximal extension of the $\mathcal{N} = 1$ super-Poincaré algebra naturally emerges from the gauge structure of the minimal extension of the AdS algebra. This means that this procedure brings all the dynamical fields required to form a connection for the maximal extension of the $\mathcal{N} = 1$ super-Poincaré algebra as well as their prescribed supersymmetric transformations that can be viewed as gauge transformations. Finally, the local supersymmetric extension of the Poincaré invariant gravity for the maximal extension of the $\mathcal{N} = 1$ super-Poincaré algebra is deduced from the supersymmetric AdS Lagrangian.

The paper is organized as follows. In the next section, we construct the local supersymmetric extension of the Poincaré invariant gravity in odd dimensions allowing Majorana spinors. We show that the resulting action can be viewed as a gauge theory with a gauge group identified with the maximal extension of the $\mathcal{N} = 1$ super-Poincaré algebra. In the third section, the same problem is addressed for the AdS invariant gravity and the link between both models is established through the expansion method. The eleven-dimensional case, because of its interest in the context of the M-theory will be reported elsewhere [15]. Finally, in the last section we summarize our results, give some comments and present some open questions.

2. Local supersymmetric extension of the Poincaré invariant gravity

We restrict ourselves to the odd dimensions allowing Majorana spinors $d = 3 + 8k$ with $k \in \mathbb{N}$. We rewrite the gravitational Lagrangian \mathcal{L}_P defined in (1.2) by using a trace expression over the Γ -matrices as

$$\mathcal{L}_P = \text{Tr} \left[\not{R}^{4k+1} \not{\epsilon} \right] \tag{2.1}$$

where we have defined

$$\not{\epsilon} = e_a \Gamma^a, \quad \not{R} = \frac{1}{2} R_{ab} \Gamma^{ab}.$$

As seen in the introduction, the Lagrangian \mathcal{L}_P possesses local Poincaré invariance and the latter, among other interesting features, can be viewed as a gauge theory for the Poincaré algebra. A natural way to construct a local supersymmetric extension of \mathcal{L}_P sharing the same features is to impose that the extra fields required by the supersymmetry also belong to a connection for some supersymmetric extension of the Poincaré algebra. In doing so, we shall see that the maximal extension of the $\mathcal{N} = 1$ super-Poincaré algebra naturally emerges in order to accommodate the extra fields required by the supersymmetry and also to prescribe their correct supersymmetric transformations. As the simplest tentative, we see whether the $\mathcal{N} = 1$ super-Poincaré algebra without central charges can accommodate this construction. In this case, the field content is just supplemented by the introduction of the gravitino ψ and the supersymmetric transformations obtained as a gauge transformation (1.4) with parameter $\lambda = \epsilon^\alpha Q_\alpha$ where ϵ is a zero-form Majorana spinor read

$$\delta e^a = (\bar{\epsilon} \Gamma^a \psi), \quad \delta \omega^{ab} = 0, \quad \delta \psi = D\epsilon := \left(d + \frac{1}{4} \omega_{ab} \Gamma^{ab} \right) \epsilon.$$

In this case, the variation of \mathcal{L}_P (2.1) under these supersymmetric transformations can be canceled by a kinetic term of the gravitino ψ given by

$$\mathcal{L}_\psi = -2^{k+1} \text{Tr} \left[\not{R}^{4k} (D\psi) \bar{\psi} \right].$$

In details, we have

$$\delta \mathcal{L}_P + \delta \mathcal{L}_\psi = \text{Tr} \left[\not{R}^{4k+1} \left((\bar{\epsilon} \Gamma_a \psi) \Gamma^a - 2^k (\epsilon \bar{\psi} - \psi \bar{\epsilon}) \right) \right],$$

and then using the following Fierz rearrangement in $d = 3 + 8k$

$$\epsilon \bar{\psi} - \psi \bar{\epsilon} = \frac{1}{2^k} (\bar{\epsilon} \Gamma_a \psi) \Gamma^a + \sum_{p \in \mathcal{P}} \frac{(-1)^{p+1}}{2^k p!} (\bar{\epsilon} \Gamma_{a_1 \dots a_p} \psi) \Gamma^{a_1 \dots a_p} \tag{2.2}$$

where the sum is over the set \mathcal{P} defined by

$$\mathcal{P} = \{ p = 2, 5 \pmod{4} \text{ with } p \leq 4k + 1 \}. \tag{2.3}$$

Finally, we obtain

$$\delta \mathcal{L}_P + \delta \mathcal{L}_\psi = - \sum_{p \in \mathcal{P}} \frac{(-1)^{p+1}}{p!} \text{Tr} \left[\not{R}^{4k+1} \left((\bar{\epsilon} \Gamma_{a_1 \dots a_p} \psi) \Gamma^{a_1 \dots a_p} \right) \right]. \tag{2.4}$$

Hence, we conclude that the standard super-Poincaré algebra is not rich enough to ensure the off-shell supersymmetry of the action. Nevertheless, it is simple to see that the variation (2.4) can be canceled by introducing bosonic one-form fields that are tensors of rank p , $b_{(p)}^{a_1 \dots a_p}$ with $p \in \mathcal{P}$, and transform as $\bar{\epsilon} \Gamma_{a_1 \dots a_p} \psi$ under supersymmetry. Assuming that these extra fields belong to a single connection, the only option is to consider an extension of the Poincaré algebra spanned by the following set of generators

$$\{J_{ab}, P_a, Q_\alpha, (Z_{a_1 \dots a_p})_{p \in \mathcal{P}}\} \quad (2.5)$$

where Q_α is the Majorana generator and the generators $(Z_{a_1 \dots a_p})_{p \in \mathcal{P}}$ are Lorentz tensors of rank p . In this case, the corresponding super connection is given by

$$A = \frac{1}{2} \omega^{ab} J_{ab} + e^a P_a + \psi^\alpha Q_\alpha + \sum_{p \in \mathcal{P}} \frac{1}{p!} b_{(p)}^{a_1 \dots a_p} Z_{a_1 \dots a_p}. \quad (2.6)$$

In addition, in order to prescribe the correct gauge supersymmetric transformations of the extra bosonic fields, the anticommutator of the Majorana generator must be given by

$$\{Q, Q\} = (C\Gamma^a)P_a + \sum_{p \in \mathcal{P}} \frac{1}{p!} (C\Gamma^{a_1 \dots a_p})Z_{a_1 \dots a_p}, \quad (2.7)$$

where the sum is over the set \mathcal{P} defined previously (2.3) and where C is the antisymmetric charge conjugation matrix. The algebra (2.7) is known as the maximal extension of the $\mathcal{N} = 1$ super-Poincaré algebra. This algebra is said maximal since the left hand side is a $2^{[d/2]} \times 2^{[d/2]}$ real symmetric matrix, so the maximal number of algebraically distinct charges that can appear on the right hand side is $2^{[d/2]} \times (2^{[d/2]} + 1)/2$, which is precisely the number of components of P_a and the different p-form “central charges” that appear in the right hand side. In eleven dimensions, this algebra is commonly known as the M-algebra since it encodes many important features of the M-theory.

The supersymmetry transformations of all the dynamical fields can be read off as a gauge transformation of the connection (2.6) for the algebra (2.7), and they are given by

$$\begin{aligned} \delta e^a &= (\bar{\epsilon} \Gamma^a \psi), & \delta \psi &= D\epsilon \\ \delta \omega^{ab} &= 0, & \delta b_{(p)}^{a_1 \dots a_p} &= (\bar{\epsilon} \Gamma^{a_1 \dots a_p} \psi). \end{aligned} \quad (2.8)$$

Finally, the local supersymmetric extension of the Poincaré invariant gravity Lagrangian in the odd dimensions $d = 3 + 8k$ is found to be

$$\mathcal{L}_P^{\text{susy}} = \text{Tr} \left[\mathcal{R}^{4k} \left(\mathcal{R} \left(\not{\epsilon} + \sum_{p \in \mathcal{P}} (-1)^{p+1} \not{b}_{(p)} \right) - (D\psi) \bar{\psi} \right) \right], \quad (2.9)$$

where we have defined

$$\not{\epsilon} = e_a \Gamma^a, \quad \mathcal{R} = \frac{1}{2} R_{ab} \Gamma^{ab}, \quad \not{b}_{(p)} = \frac{1}{p!} b_{(p)}^{a_1 \dots a_p} \Gamma_{a_1 \dots a_p}.$$

The invariance of (2.9) with respect to the supersymmetric transformations (2.8) can easily be checked with the use of the Fierz rearrangement (2.2).

Hence, in odd dimensions allowing Majorana spinors, a supersymmetric extension of the Poincaré invariant gravity can be constructed for which the resulting action is a gauge theory for the maximal extension of the $\mathcal{N} = 1$ super-Poincaré algebra.

3. Local supersymmetric extensions of the AdS invariant gravity

In this section, we shall be concerned with the supersymmetric extension of the AdS invariant gravity in odd dimensions allowing Majorana spinors, $d = 3 + 8k$. We shall also establish a connection between this theory and the Poincaré one derived previously at some vanishing cosmological constant limit.

In odd dimensions, the Lorentz symmetry of the Lovelock theory (1.1) can also be extended to a local AdS symmetry and the resulting Lagrangian is given by

$$\mathcal{L}_{\text{AdS}}^{(2n+1)} = \sum_{q=0}^n \frac{\binom{n}{q}}{(2n+1-2q)} \mathcal{L}^{(q)}, \quad (3.1)$$

where the Lagrangians $\mathcal{L}^{(q)}$ are defined by

$$\mathcal{L}^{(q)} = \epsilon_{a_1 \dots a_d} R^{a_1 a_2} \dots R^{a_{2q-1} a_{2q}} e^{a_{2q+1}} \dots e^{a_d}. \quad (3.2)$$

The action (3.1) defines a $(2n+1)$ -Chern-Simons form of the AdS group. The supersymmetric extensions of the AdS gravity action (3.1) have been constructed in three [16, 17], five [18] and higher odd dimensions [19].

In the present case, we are concerned with the odd dimensions allowing Majorana spinors, $d = 3 + 8k$. For these particular dimensions, van Holten and Von Proeyen have derived the minimal super algebras that contain the AdS generators by adding one Majorana supersymmetry generator and by demanding the closure of the full super algebra [20]. In particular, the consequences of the $[P, Q, Q]$ Jacobi identity imply that the anticommutator of the Majorana generator is given by

$$\{Q, Q\} = (C\Gamma^a)P_a - \frac{1}{2}(C\Gamma^{ab})J_{ab} + \sum_{p' \in \mathcal{P}'} \frac{1}{p'!} (C\Gamma^{a_1 \dots a_{p'}}) Z_{a_1 \dots a_{p'}}, \quad (3.3)$$

where the sum is over the set \mathcal{P}' defined by

$$\mathcal{P}' = \{p' = 5, 6 \pmod{4} \text{ with } p' \leq 4k + 1\}. \quad (3.4)$$

We insist on the notation p' to stress the difference with the extended super-Poincaré algebras (2.7) where the membrane value $p = 2$ is included. The super algebras described by the anticommutation relation (3.3) is known as the orthosymplectic group $\text{Osp}(2^{[d/2]}|1)$. There exist important differences between the minimal super AdS algebras (3.3) and the maximal extensions of the super-Poincaré algebras (2.7). For a fixed dimension d , the super algebra $\text{Osp}(2^{[d/2]}|1)$ has $d(d-1)/2$ less generators than the algebra (2.7) owing to the fact that the Poincaré “central charge” Z_{ab} is not a generator of the super AdS algebra. Another important difference is the presence of the Lorentz generator J_{ab} in the right hand side of the anticommutation relation (3.3) which in turns implies that the supersymmetric AdS transformation of the spin connection does not vanish (3.6).

The supersymmetric extension of the AdS gravity (3.1) in dimension $d = 3 + 8k$ can be constructed as follows. We define a connection one-form A in the adjoint representation of $\text{Osp}(2^{[d/2]}|1)$. Then we construct the Chern-Simons form associated as follows:

$$d\mathcal{L}_{\text{AdS}}^{\text{susy}} = \text{STr} \left(F^{4k+2} \right), \quad (3.5)$$

where ‘‘STr’’ stands for the super trace and F is the curvature associated to the connection A . The supersymmetric transformations read off as gauge transformations are given by

$$\begin{aligned} \delta e^a &= (\bar{\epsilon} \Gamma^a \psi), & \delta \psi &= \nabla \epsilon \\ \delta \omega^{ab} &= -(\bar{\epsilon} \Gamma^{ab} \psi), & \delta b_{(p')}^{a_1 \dots a_{p'}} &= (\bar{\epsilon} \Gamma^{a_1 \dots a_{p'}} \psi) \end{aligned} \quad (3.6)$$

where the covariant derivative now reads

$$\nabla \epsilon = D\epsilon + \left(e_a \Gamma^a + \sum_{p'} b_{(p')}^{a_1 \dots a_{p'}} \Gamma_{a_1 \dots a_{p'}} \right) \epsilon.$$

As we are dealing with a theory in presence of a negative cosmological constant Λ , a natural question to ask is whether the limiting case $\Lambda \rightarrow 0$ yields to interesting features. Moreover, it is well-known that the Poincaré algebra can be viewed as a Wigner-Inönü contraction of the (A)dS algebra, and so it is legitimate to ask whether the Poincaré supersymmetric theories described in the previous section can be derived from the super AdS ones at the zero cosmological constant limit. In eleven dimensions, this problem has been considered in [15] where a generalization of the Wigner-Inönü contraction has permitted to link the AdS supersymmetric theory invariant under $\text{Osp}(32|1)$ with a gauge theory for the M-algebra. The aim of this section is to show that the conclusions obtained in eleven dimensions are still valid for all the odd dimensions admitting Majorana spinors. More precisely, we first show that the maximal Poincaré algebras (2.7) can be obtained from the minimal super AdS algebras (3.3) through the expansion method that permits to generate Lie algebra of higher dimensions [12]. In addition, we shall see that the implementation of this expansion on the dynamical fields also permits to derive the supersymmetric extension of the Poincaré invariant gravity (2.9) from the minimal super AdS theory (3.5).

In order to point out the necessity of considering the expansion method rather than a standard Wigner-Inönü contraction, we first operate a standard Wigner-Inönü contraction on the supersymmetric AdS action (3.5). The implementation of this contraction on the AdS dynamical fields is given as usual by

$$e^a \rightarrow \frac{1}{l} e^a, \quad \omega^{ab} \rightarrow \omega^{ab}, \quad b_{(p')}^{a_1 \dots a_{p'}} \rightarrow \frac{1}{l} b_{(p')}^{a_1 \dots a_{p'}}, \quad \psi \rightarrow \frac{1}{\sqrt{l}} \psi, \quad (3.7)$$

where l is a scaling parameter for the radius of the universe and the zero cosmological constant limit corresponds to taking $l \rightarrow \infty$ and where p' run over the set \mathcal{P}' defined in (3.4). On the other hand, since the gauge parameter ϵ of the supersymmetric transformations must also be rescaled as $\epsilon \rightarrow 1/\sqrt{l} \epsilon$, the supersymmetric transformations $\bar{\delta}$ defined by (3.6) reduce to those associated to the extended super-Poincaré algebra (2.8) with the exception that the bosonic field $b_{(2)}^{ab}$ is not present in the AdS theory (3.4). Operating the rescaling (3.7) at the level of the supersymmetric action (3.5), the latter can be expanded as follows

$$\begin{aligned} \mathcal{L}_{\text{AdS}}^{\text{susy}} &= \mathcal{L}^*(\omega) + \frac{1}{l} \text{Tr} \left[\mathcal{R}^{4k+1} \left(\not{\epsilon} + \sum_{p' \in \mathcal{P}'} (-1)^{p'+1} b_{(p')} - \mathcal{R}^{4k} (D\psi) \bar{\psi} \right) \right] + o(l^{-2}), \\ &= \mathcal{L}^{(0)} + \frac{1}{l} \mathcal{L}^{(1)} + o(l^{-2}) \end{aligned} \quad (3.8)$$

where $\mathcal{L}^{(0)} = \mathcal{L}^*(\omega)$ is the Lorentz Chern-Simons form which depends only on the spin connection and which satisfies

$$d\mathcal{L}^*(\omega) = \text{Tr} \left(\mathbb{R}^{A_{k+2}} \right). \quad (3.9)$$

It is clear that in the limit $l \rightarrow \infty$, the supersymmetric Lagrangian $\mathcal{L}_{\text{AdS}}^{\text{susy}}$ reduces to the Lorentz Chern-Simons form which is trivially supersymmetric with respect to the Poincaré supersymmetric transformations (2.8) since it depends only on the spin connection and $\delta\omega^{ab} = 0$. This means that although the Wigner-Inönü contraction gives rise to a consistent theory, the resulting Lagrangian, being decoupled of the vielbein and the gravitino, is of little interest. The next order l^{-1} in the expansion (3.8) is more interesting since it contains the Poincaré invariant gravity Lagrangian. However, it is clear with the use of the Fierz rearrangement (2.2) that the expression at the order l^{-1} is not supersymmetric because of the absence of the bosonic one-form field $b_{(2)}^{ab}$. The natural way to introduce this field is to exploit the fact that one can always add to the spin connection a tensor under the Lorentz group as

$$\omega^{ab} \rightarrow \omega^{ab} - \frac{1}{l} b_{(2)}^{ab}. \quad (3.10)$$

Apart from introducing the required bosonic field $b_{(2)}^{ab}$, the splitting (3.10) has two other important consequences. Firstly, it prescribes the correct supersymmetric transformation of the bosonic field $b_{(2)}^{ab}$,

$$\delta\omega^{ab} - \frac{1}{l} \delta b_{(2)}^{ab} = -\frac{1}{l} \left(\bar{\epsilon} \Gamma^{ab} \psi \right) \implies \delta\omega^{ab} = 0, \quad \delta b_{(2)}^{ab} = \left(\bar{\epsilon} \Gamma^{ab} \psi \right). \quad (3.11)$$

Secondly, the Lorentz Chern-Simons form \mathcal{L}^* brings now a contribution at the order l^{-1} in the expansion given by

$$\mathcal{L}^*\left(\omega - \frac{1}{l} b_{(2)}\right) = \mathcal{L}^*(\omega) - \frac{1}{l} \text{Tr} \left(\mathbb{R}^{A_{k+1}} \mathbb{b}_{(2)} \right) + o(l^{-2}).$$

Combining this expression together with (3.8) shows that the expression at the order l^{-1} in the expansion becomes precisely the supersymmetric action associated to the maximal super-Poincaré algebra (2.9).

Hence, the connection between the AdS and Poincaré supersymmetric theories has been realized through a standard Wigner-Inönü contraction supplemented by a splitting of the spin connection. These are the two basic ingredients of the expansion method at the algebraic level which is described as follows. Firstly, one trivially extend the super AdS algebra with Lorentz generators T_{ab} satisfying $[T_{ab}, T_{cd}] = -T_{ac}\eta_{bd} + \dots$ and, secondly one perform the following contraction

$$J_{ab} \rightarrow J_{ab} - T_{ab}, \quad Z_{ab} = \frac{1}{l} T_{ab}, \quad P_a \rightarrow l P_a, \quad Z_{a_1 \dots a_{p'}} \rightarrow l Z_{a_1 \dots a_{p'}}, \quad Q \rightarrow \sqrt{l} Q, \quad (3.12)$$

where l is the parameter of the expansion. In the limit $l \rightarrow \infty$, the minimal extension of the AdS algebra (3.3) expressed in terms of the generators $J_{ab}, P_a, Z_{ab}, Z_{a_1 \dots a_{p'}}$ and Q_α

becomes precisely the maximal supersymmetric extension of the Poincaré algebra (2.7). The first operation is compatible with the splitting of the spin connection while the second operation is nothing but a standard Wigner-Inönü contraction.

The lack of supersymmetry of the Lagrangian $\mathcal{L}^{(1)}$ in the expression (3.8) is due to the presence of the Lorentz-Chern-Simons form. This form defined only in odd dimensions $d = 4k - 1$ (which includes the odd dimensions allowing Majorana spinors) is part of the Pontryagin-Chern-Simons form which is required in order to further extend the AdS symmetry into supersymmetry without duplicating the field content of the theory [21]. Indeed, the action of the standard Wigner-Inönü contraction on the dynamical fields (3.7) also affect the original supersymmetric transformations (3.6). Indeed, the gauge parameter ϵ must be rescaled as $\epsilon \rightarrow 1/\sqrt{l} \epsilon$ and, as a consequence the supersymmetric transformations (3.6) are split into two different orders. In particular the variation of the spin connection gets a contribution of the first order,

$$\delta\omega^{ab} = -\frac{1}{l} (\bar{\epsilon}\Gamma^{ab}\psi) \implies \delta^{(0)}\omega^{ab} = 0, \quad \delta^{(1)}\omega^{ab} = -\frac{1}{l} (\bar{\epsilon}\Gamma^{ab}\psi) \quad (3.13)$$

while for the remaining dynamical fields we have

$$\begin{aligned} \delta^{(0)}e^a &= (\bar{\epsilon}\Gamma^a\psi), & \delta^{(1)}e^a &= 0 \\ \delta^{(0)}b_{(p')}^{a_1\dots a_{p'}} &= (\bar{\epsilon}\Gamma^{a_1\dots a_{p'}}\psi), & \delta^{(1)}b_{(p')}^{a_1\dots a_{p'}} &= 0 \\ \delta^{(0)}\psi &= D(\omega)\epsilon, & \delta^{(1)}\psi &= \left(e_a\Gamma^a + \sum_{p'} b_{(p')}^{a_1\dots a_{p'}} \Gamma_{a_1\dots a_{p'}} \right) \epsilon \end{aligned} \quad (3.14)$$

Hence the variation of the original supersymmetric Lagrangian (3.8) under the supersymmetric transformations (3.6) gives a series in powers of l^{-1} in which each order is a total derivative, i.e.

$$\begin{aligned} \delta\mathcal{L}_{AdS}^{susy} &= [\delta^{(0)}\mathcal{L}^{(0)}] + \frac{1}{l}[\delta^{(0)}\mathcal{L}^{(1)} + \delta^{(1)}\mathcal{L}^{(0)}] + o(l^{-2}) \\ &= d\Sigma_0 + \frac{1}{l}d\Sigma_1 + o(l^{-2}). \end{aligned} \quad (3.15)$$

From this expression, it is clear that the Lagrangian $\mathcal{L}^{(1)}$ is not supersymmetric w. r. t. the transformations $\delta^{(0)}$ since the variation of the Lorentz-Chern-Simons form $\delta^{(1)}\mathcal{L}^{(0)}$ does not vanish and is not a surface term. This is because $\mathcal{L}^{(0)}$ depends on the spin connection and the variation of the latter has a contribution at the first order (3.13). To circumvent this problem, it would be sufficient if the variation $\delta^{(1)}\omega^{ab}$ vanished identically, and this is precisely what we have done previously. It is interesting to note that in doing so, the transformations $\delta^{(0)}$ given in (3.14) together with the transformation of the extra field arising from the splitting of the spin connection (3.11) are precisely the gauge transformations associated to the maximal extension of the $\mathcal{N} = 1$ super-Poincaré algebra (2.8). Hence the expansion method in order to make supersymmetric the next order naturally brings in the gauge structure of the maximal extension of the $\mathcal{N} = 1$ super-Poincaré algebra.

In this analysis, we have been concerned by the $(4k + 2)$ th Chern character whose potential Chern-Simons is given by the Lagrangian \mathcal{L}_{AdS}^{susy} , (3.5). However, in dimension

$d = 3 + 4k$, there exist many other Chern character forms such that $\text{STr}(F^{4k}) \text{STr}(F^2)$ or $[\text{STr}(F^2)]^{1+2k}$. A natural question to ask is whether our conclusions depend on the choice of the Chern character. However, it is tedious but straightforward to see that any linear combination of all the Chern characters, namely

$$\alpha_1 \text{STr}(F^{4k+2}) + \alpha_2 \text{STr}(F^{4k}) \text{STr}(F^2) + \dots + \alpha_p [\text{STr}(F^2)]^{1+2k},$$

will lead through the same procedure to the same conclusions: the Wigner-Inönü contraction will give a Lagrangian decoupled from the vielbein and in order to make the next order supersymmetric, the maximal extension of the $\mathcal{N} = 1$ super-Poincaré algebra will naturally emerge. The resulting Lagrangian will be the one derived in the previous section (2.9) up to some additional terms decoupled from the vielbein that are supersymmetric by themselves.

4. Discussion

In odd dimensions allowing Majorana spinors $d = 3 + 8k$, we have considered a gravitational Lagrangian given by the dimensional continuation of the Euler density and possessing the local Poincaré invariance. We have constructed the local supersymmetric extension of this Poincaré invariant gravity Lagrangian and we have shown that the resulting action can be viewed as a gauge theory for the maximal extension of the $\mathcal{N} = 1$ super-Poincaré algebra. We have addressed the same question for the AdS invariant gravity for which we have constructed its local supersymmetric extension for the minimal extension of the AdS algebra. In these particular dimensions, the maximal super-Poincaré algebra has more generators than the minimal super AdS algebra, and hence it is clear that these two super algebras can not be linked through a process like the Wigner-Inönü contraction that does not increase or decrease the number of generators. This is the reason for which we have taken advantage of the expansion method developed in [12] which permits to link Lie algebras of different dimensions. More precisely, the maximal extension of the $\mathcal{N} = 1$ super-Poincaré algebra can be obtained through the expansion from the minimal super AdS algebra. This correspondence at the algebraic level has been extended at the level of the corresponding supersymmetric actions. In fact, we have obtained the local supersymmetric extension of this Poincaré invariant gravity Lagrangian from the minimal super AdS gravity Lagrangian by a process compatible with the expansion method. In doing so, we have pointed out that the necessity of expanding the minimal super AdS algebra rather than considering a standard Wigner-Inönü is a direct consequence of the presence of the Lorentz-Chern-Simons form (3.9). This form defined only in odd dimensions $d = 4k - 1$ is part of the Pontryagin-Chern-Simons form which is required in order to further extend the AdS symmetry into supersymmetry without duplicating the field content of the theory [21]. It would be interesting to see what is the correct correlation between the presence of the Pontryagin-Chern-Simons form and the necessity of expanding the algebra rather than operating a standard contraction. We have also shown that our conclusions do not depend on the choice of the Chern character and, any linear combination of the different Chern characters will lead to the same conclusion.

In $d = 3 + 8k$, the expansion has been realized at the first order. This has permitted to supersymmetrize the Poincaré invariant gravity Lagrangian. At the order $1 + 8k$, it will appear the standard Einstein-Hilbert Lagrangian and hence, it will be interesting to see whether one can construct a consistent supersymmetric theory at this order through the same procedure. Note that starting from the AdS invariant gravity Lagrangian in the absence of supersymmetry, it is possible to deform the theory through the expansion of the Lie algebra and get a system consisting of the Einstein-Hilbert action plus nonminimally coupled matter [22].

In the expansion procedure, we have pointed out that the resulting supersymmetric extension of the Poincaré invariant gravity is a gauge theory for the maximal extension of the super-Poincaré algebra. However, there exist supersymmetric extensions of the Poincaré invariant gravity that are gauge systems for some subalgebras of the maximal extension of the super-Poincaré algebra. It is clear from our analysis that these theories can not be reached through the expansion. It will be interesting to explore if there exists other process that will realize this task. In some case, the subalgebra of the maximal extension of the super-Poincaré algebra has the same dimension than the minimal super AdS algebra. However, although these algebras can be put in correspondence through the Wigner-Inönü contraction, this is not true for the corresponding supersymmetric actions. Hence, it would be interesting to determine whether generalizations of the Wigner-Inönü contraction described in [23] or in [24] can be useful for generating other supersymmetric theories than those obtained through the expansion method.

Finally, we have only been concerned with the odd dimensions allowing Majorana spinors. A natural extension of this work will consist in considering all the odd dimensions and to realize an exhaustive analysis [25].

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References

- [1] C. Lanczos, *A Remarkable property of the Riemann-Christoffel tensor in four dimensions*, *Ann. Math.* **39** (1938) 842.
- [2] D. Lovelock, *The Einstein tensor and its generalizations*, *J. Math. Phys.* **12** (1971) 498.
- [3] C. Teitelboim and J. Zanelli, *Dimensionally continued topological gravitation theory in Hamiltonian form*, *Class. Quant. Grav.* **4** (1987) L125.
- [4] B. Zwiebach, *Curvature squared terms and string theories*, *Phys. Lett.* **B 156** (1985) 315.
- [5] N. Marcus and J.H. Schwarz, *Three-dimensional supergravity theories*, *Nucl. Phys.* **B 228** (1983) 145;
S. Deser and J.H. Kay, *Topologically massive supergravity*, *Phys. Lett.* **B 120** (1983) 97;

- S. Deser, *Topologically massive gravity: a renormalizable theory*, in *Quantum theory of gravity: essays in honor of the 60th birthday of Bryce S. deWitt*, S.M. Christensen ed., Adam Hilger, London, U.K. (1984).
- [6] M. Bañados, R. Troncoso and J. Zanelli, *Higher dimensional Chern-Simons supergravity*, *Phys. Rev. D* **54** (1996) 2605 [[gr-qc/9601003](#)].
- [7] J. Zanelli, *Lecture notes on Chern-Simons (super-)gravities*, [hep-th/0502193](#); *Chern-Simons gravity: from 2 + 1 to 2n + 1 dimensions*, *Braz. J. Phys.* **30** (2000) 251; J.D. Edelstein and J. Zanelli, *(Super-)Gravities of a different sort*, *J. Phys. Conf. Ser.* **33** (2006) 83 [[hep-th/0605186](#)].
- [8] M. Hassaïne, R. Olea and R. Troncoso, *New N = 2 supergravity with local Poincaré invariance in nine dimensions*, *Phys. Lett. B* **599** (2004) 111 [[hep-th/0210116](#)].
- [9] M. Hassaïne, R. Troncoso and J. Zanelli, *Eleven-dimensional supergravity as a gauge theory for the M-algebra*, *Phys. Lett. B* **596** (2004) 132 [[hep-th/0306258](#)]; *11D Supergravity as a gauge theory for the M-algebra*, *PoS(WC2004)006* [[hep-th/0503220](#)].
- [10] E. Cremmer, B. Julia and J. Scherk, *Supergravity theory in 11 dimensions*, *Phys. Lett. B* **76** (1978) 409.
- [11] A. Van Proeyen, *Tools for supersymmetry*, [hep-th/9910030](#).
- [12] J.A. de Azcarraga, J.M. Izquierdo, M. Picón and O. Varela, *Generating Lie and gauge free differential (super)algebras by expanding Maurer-Cartan forms and Chern-Simons supergravity*, *Nucl. Phys. B* **662** (2003) 185 [[hep-th/0212347](#)]; *Expansions of algebras and superalgebras and some applications*, *Int. J. Theor. Phys.* **46** (2007) 2738 [[hep-th/0703017](#)].
- [13] M. Hatsuda and M. Sakaguchi, *Wess-Zumino term for the AdS superstring and generalized Inoenue-Wigner contraction*, *Prog. Theor. Phys.* **109** (2003) 853 [[hep-th/0106114](#)].
- [14] I.E. Segal, *A class of operator algebras which are determined by groups*, *Duke Math. J.* **18** (1951) 221.
- [15] M. Hassaïne, R. Troncoso and J. Zanelli, *Emergence of the M-algebra from eleven-dimensional AdS supergravity*, CECS-PHY-06/08.
- [16] A. Achúcarro and P.K. Townsend, *A Chern-Simons Action for three-dimensional Anti-de Sitter supergravity theories*, *Phys. Lett. B* **180** (1986) 89.
- [17] A. Giacomini, R. Troncoso and S. Willison, *Three-dimensional supergravity reloaded*, *Class. and Quant. Grav.* **24** (2007) 2845 [[hep-th/0610077](#)].
- [18] A.H. Chamseddine, *Topological gravity and supergravity in various dimensions*, *Nucl. Phys. B* **346** (1990) 213.
- [19] R. Troncoso and J. Zanelli, *New gauge supergravity in seven and eleven dimensions*, *Phys. Rev. D* **58** (1998) 101703 [[hep-th/9710180](#)]; *Gauge supergravities for all odd dimensions*, *Int. J. Theor. Phys.* **38** (1999) 1181 [[hep-th/9807029](#)]; *Higher dimensional gravity and local AdS symmetry*, *Class. and Quant. Grav.* **17** (2000) 4451 [[hep-th/9907109](#)].
- [20] J.W. van Holten and A. Van Proeyen, *N = 1 Supersymmetry algebras in D = 2, D = 3, D = 4 MOD-8*, *J. Phys. A* **15** (1982) 3763.
- [21] P. Hořava, *M-theory as a holographic field theory*, *Phys. Rev. D* **59** (1999) 046004 [[hep-th/9712130](#)].

- [22] J.D. Edelstein, M. Hassaine, R. Troncoso and J. Zanelli, *Lie-algebra expansions, Chern-Simons theories and the Einstein-Hilbert lagrangian*, *Phys. Lett. B* **640** (2006) 278 [[hep-th/0605174](#)].
- [23] E. Weimar-Woods, *Contractions of Lie algebras: Generalized Inönü-Wigner contractions versus graded contractions*, *J. Math. Phys.* **36** (1995) 4519; *Contractions, generalized Inönü-Wigner contractions and deformations of finite-dimensional Lie algebras*, *Rev. Math. Phys.* **12** (2000) 1505.
- [24] F. Izaurieta, E. Rodriguez and P. Salgado, *Expanding Lie (super)algebras through Abelian semigroups*, *J. Math. Phys.* **47** (2006) 123512 [[hep-th/0606215](#)].
- [25] M. Hassaine and M. Romo, work in progress.